FOUR DRAFTS OF THE REPRESENTATION THEORY OF THE GROUP OF INFINITE MATRICES OVER A FINITE FIELD

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Preface 2007. About these texts

The four texts presented below were written in 1997–2000, when S. V. Kerov and me decided to return to the program we had designed in the early 80s: to generalize the representation theory of the infinite symmetric group, which we had started in the late 70s, to the groups of infinite matrices over finite fields. Our first basic publication on this subject [VK] was unfortunately almost the last at that time — the only later publication was my two introductory lectures published in the Proceedings of the EMS Summer School [V] (the second lecture was devoted to this topic). The four notes that are presented below have not been completely finished and are not yet published. After Sergey's untimely death in July 2000, I did not continue to work on this project, only gave several talks in about 2001–2002 on our preliminary results. Now I have decided to keep them in the original form, except for adding a few additional remarks, corrections, and references,[1] so this is a kind of prepublication, whose goal is to stimulate and continue the study of the subject which I consider extremely important, especially nowadays, because of the increasing interest of mathematicians to various q-analogs of classical objects.

It makes sense to read these drafts after the paper [VK], but we will not reproduce it here. Nevertheless, we recall some important facts that will help to join these drafts together.

It is well known that the main object that can be considered as a q-analog of the symmetric group $\mathfrak{S}_n$ is the group $GL(q,n)$ of matrices over the finite field $\mathbb{F}_q$ (with q a power of a prime). But it turned out that for the infinite symmetric group $\mathfrak{S}_\infty$, the right q-analog is not the group $GL(q,\infty)$, which is the inductive limit of the groups $GL(q,n)$ in n, but another group, which we denoted by $GLB(\mathbb{F}_q)$; it is neither inductive nor projective limit of finite groups, but a so-called IP-limit (inductive-projective limit), see below. This is the locally compact group of all infinite matrices over the finite field that are finite (= contain finitely many nonzero elements) below the main diagonal. Thus the Borel subgroup of this group is compact. At the same time, one version of the group algebra of the group GLB (an analog of the Bruhat–Schwartz algebra of the classical p-adic groups) is the inductive limit of the group algebras $\mathbb{C}(GL(q,n))$, so that we can apply the methods of inductive theory. This inductive limit of algebras does not correspond to any inductive limit of the corresponding groups; at the group level, this means that we use a more complicated construction, which is described here. The reason why the

[1] The list of references given at the end of the fourth draft is common for all four drafts and contains only strictly necessary references.

group GLB is the right infinite q-analog of $\mathfrak{S}_\infty$ is related to the fundamental fact of the classical representation theory of the groups $GL(q,n)$: if we want the restrictions of irreducible representations of $GL(q,n+1)$ to $GL(q,n)$ to be multiplicity-free, we must use the *parabolic embedding* of $GL(q,n)$ into $GL(q,n+1)$ (see [Green, Fad, Zel]). More exactly, the natural procedure for extending a representation of $GL(q,n)$ to a representation of $GL(q,n+1)$ is not the usual induction from $GL(q,n)$ as a subgroup of $GL(q,n+1)$, but the induction from a maximal parabolic subgroup of $GL(q,n+1)$ that contain $GL(q,n)$. Thus we need to respect this rule for embedding and generalize it to the infinite case.[2]

In the first draft presented below ("Interpolation between inductive and projective limits of finite groups with applications to linear groups over finite fields") we consider a general scheme called the IP-limit of groups, the main example of which is the group GLB as the IP-limit of $GL(q,n)$. It is very interesting to find other nontrivial examples of this scheme. It is worth adding that the idea of the mixed (indo-projective, or IP) limit was used in several categories (convex sets, linear spaces, etc.). The first author used it for studying the notion of the tower of measures (see [V94]). Here we consider the pure group-theoretical aspect of this notion.

The next note, "The characters of the group of almost triangular matrices over a finite field," is devoted to the main topic, the theory of characters of the groups under consideration. First of all, at the first stage it is better to restrict ourselves to the so-called principal series of representations and unipotent characters. It is worth mentioning that [VK] contains a mistake, which was found after recent discussions with E. Goryachko. Namely, the Bratteli diagram for the group GLB is not the disjoint union of copies of the Young graph, as we claimed in [VK]; fortunately, this claim does not affect any other assertion in [VK]. The diagram in question is indeed the disjoint union of countably many components, but each of these components is the direct product of copies of the Young graph (depending on the field). Thus the principal series is only a part of the main component of the Bratteli diagram; nevertheless, the branching rule and the structure of characters of all other series (caspidal etc.) are almost the same as for the principal series. That is why we switch on from the group GLB to the subgroup $GLN(q)$, which is the IP-limit with the "unipotent" projections; the difference between these groups is merely that the diagonal elements of matrices in GLN are eventually 1's and the maximal unipotent subgroup consists of upper triangular matrices with 1's on the diagonal.

We formulate a series of conjectures about the unipotent characters of the group GLN, and the main fact is (see also the third draft) that every indecomposable character determines on the Borel (A. Borel!) subgroup *a true Borel* (E. Borel!) probability measure, and these measures are mutually singular for different characters.

In the next draft, "A Law of Large Numbers for the characters of $GL_n(k)$ over a finite field k," which partially reproduces the previous one, we try to suggest a program of how to connect the theory of characters with the structure of Jordan forms of typical matrices (in the sense of the above-mentioned measures). This note concerns a deep and very intriguing link between two appearances of the

[2]As mentioned in [VK], the notion of the group GLB was suggested by the authors and A. Zelevinsky in 1983.

Thoma parameters (see [Thoma]): in the first case they arise in the representation theory of the infinite symmetric group and the Hecke algebras as the frequences of rows and columns of Young diagrams corresponding to representations; and in the second case they arise as the normalized lengths of Jordan blocks of matrices. This is a manifestation of some kind of duality, which also appeared in our works on models of factor representations of the infinite symmetric group. In our theory the characters are realized as the traces of well-defined type II_∞ representations. Note that this is perhaps the first occasion when an infinite trace on the group algebra of a locally finite group can be interpreted not as a function on the group and not as a degenerate linear functional on the group algebra, but as a singular (with respect to the Haar measure) measure on the group; moreover, different characters generate mutually singular measures, and for most characters the measures are supported by the whole group.

The last draft is devoted to the problem concerning realizations of representations, which is most important for applications. We give the simplest example of an implicit construction of principal series representations of the group GLB, which is based on the geometry of the infinite-dimensional Grassmanian; from the point of view of Schubert cells, this construction corresponds to infinite Young diagrams with two rows. One of the puzzles here is how a type III factor representation comes to the picture. It seems to me that construction of factor representations and some irreducible representations of GLB-type groups is the mainstream of the forthcoming research in this area, which will be continued.

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A. Vershik

I. INTERPOLATING BETWEEN INDUCTIVE AND PROJECTIVE LIMITS OF FINITE GROUPS WITH APPLICATIONS TO LINEAR GROUPS OVER FINITE FIELDS

1. IP-families of finite groups. In this section we introduce IP-families, generalizing both inductive and projective families of finite groups.

Definition. Consider a system

$$\{1\} = P_1 \subset G_1 \leftarrow P_2 \subset G_2 \leftarrow P_3 \subset G_3 \leftarrow \dots$$

of finite groups and group homomorphisms, and assume that all maps $\pi : P_{m+1} \to G_m$ are epimorphisms. We refer to such systems as *IP-families.*

Example 1. Every inductive family

$$G_1 \subset G_2 \subset G_3 \subset \dots$$

determines a trivial IP-family with $P_{m+1} = G_m$.

Example 2. Every projective family

$$G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \dots$$

determines a trivial IP-family with $P_m = G_m$.

The following two examples are less trivial and motivate our definition of IP-families.

Example 3. Let $k = \mathbb{F}_q$ be the finite field with q elements and $G_m = GL_m(k)$ the general linear group over k. We take for $P_{m+1} \subset G_{m+1}$ the subgroup of block matrices of the form

$$g = \begin{pmatrix} A & b \\ 0 & a \end{pmatrix}$$

where $A \in G_m$, $a \in G_1 \cong k^*$, and $b \in k^m$. The epimorphism $\pi : P_{m+1} \to G_m$ maps $g \in P_{m+1}$ to the matrix A. This is our basic example of an IP-family.

Example 4. The groups $G_m = GL_m(k)$ are the same as in Example 3, but we take for P_{m+1} the affine subgroup of matrices of the form

$$g = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}.$$

The epimorphisms $\pi : P_{m+1} \to G_m$ are defined as in Example 3.

Example 5. Here is a baby example intermediate between the trivial examples 1, 2 and the main examples 3, 4. Let H be a finite group, and denote by G_m the wreath product of the symmetric group $\mathfrak{S}_m$ with the coefficient group H. One can think of G_m as the group of $m \times m$ permutation matrices with nontrivial values in the group H. We take for P_{m+1} the subgroup of matrices in G_{m+1} with a nontrivial element on the crossing of the last row and the last column (so that the other elements of the last row and column vanish). The value of the epimorphism π at a matrix $g \in P_{m+1}$ is, by definition, the submatrix of g on the crossing of the first m rows and columns.

2. Associated limiting groups. In this section we define the limiting group of an IP-family, and its basic compact subgroup.

We start with a definition of subgroups $G(m,n) \subset G_n$, for all $1 \leq m \leq n$. To this end we set $G(m,m) = G_m$, and we define $G(m,n+1)$ by induction as the preimage $\pi^{-1}\big(G(m,n)\big)$ of the group $G(m,n) \subset G_n$ in $P_{n+1} \subset G_{n+1}$. In particular, $G(m,m+1) = P_{m+1}$.

The restrictions of the homomorphisms π to the subgroups $G(m,n) \subset P_n$ determine epimorphisms $\pi : G(m,n+1) \to G(m,n)$, so that a projective family

$$G(m,m) \leftarrow G(m,m+1) \leftarrow G(m,m+2) \leftarrow \dots$$

arises. We denote by $G(m,\infty) = \varprojlim_n G(m,n)$ the corresponding profinite group. By definition, an element $g \in G(m,\infty)$ is a sequence $g = \{g_n\}_{n=m}^{\infty}$ of elements $g_n \in G(m,n)$ such that $\pi(g_{n+1}) = g_n$ for all $n \geq m$.

Given an element $g = \{g_n\}_{n=m}^{\infty} \in G(m,\infty)$, the truncated sequence $\widetilde{g} = \{g_n\}_{n=m+1}^{\infty}$ determines an element $\widetilde{g}$ of the group $G(m+1,\infty)$. It follows that the groups $G(m,\infty)$ form an inductive family of compact groups

$$G(1,\infty) \subset G(2,\infty) \subset G(3,\infty) \subset \dots .$$

Definition. We denote by

$$G = \varinjlim_m \varprojlim_n G(m,n)$$

the inductive limit of the groups $G(m,\infty)$, and we consider G as the *limiting group* of an IP-family. In the inductive limit topology the group G is locally compact and totally disconnected. The compact subgroup $B = G(1,\infty) \subset G$ will be referred to as the *basic profinite subgroup* of the group G.

We denote by μ the Haar measure on the group G normalized by the condition $\mu(B) = 1$.

3. The Bruhat–Schwartz group algebra. Let $\mathcal{B}$ denote the linear space of locally constant finitely supported functions on G with complex values. The space $\mathcal{B}$ is closed under the convolution

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})\, f_2(h)\, d\mu(h), \qquad f_1, f_2 \in \mathcal{B},$$

and under the standard involution

$$f^{\#}(g) = \overline{f(g^{-1})}, \qquad f \in \mathcal{B}.$$

The functions in $\mathcal{B}$ separate the points of the group G. We use the $*$-algebra $\mathcal{B}$ as the basic group algebra of the group G.

The main advantage of this choice of $\mathcal{B}$ is that this algebra is locally semisimple, hence it can be studied by the powerful combinatorial techniques of Bratteli diagrams (see [LSS]).

Let $\mathcal{B}_m$ be the $*$-subalgebra of those functions in $\mathcal{B}$ that are supported by the subgroup $G(m,\infty) \subset G$ and does not depend but on the image $g_m \in G_m$ of an element $g = \{g_n\}_{n=m}^{\infty} \in G(m,\infty)$. The algebra $\mathcal{B}_m$ is clearly $*$-isomorphic to the

group algebra $\mathbb{C}[G_m]$. Let $\widetilde{b}_g \in \mathcal{B}_m$ denote the characteristic function of the cylinder subset $\mathrm{Cyl}(g) \subset G(m, \infty)$, where

$$\mathrm{Cyl}(g) = \{g = \{g_n\}_{n=m}^{\infty} \in G(m, \infty) :\ g_m = g\}, \qquad g \in G_m.$$

We also set $b_g = \widetilde{b}_g / |N_m|$, where N_m is the (finite) kernel of the epimorphism $\pi :\ P_{m+1} \to G_m$. Then

$$b_g * b_h = b_{gh}, \qquad (b_g)^{\#} = b_{g^{-1}}, \qquad g, h \in G_m.$$

If E_m denotes the identity element of the group G_m, then b_{E_m} is the identity in the algebra $\mathcal{B}_m$.

Proposition. *The algebra $\mathcal{B}$ is locally semisimple. More precisely, $\mathcal{B} = \varinjlim \mathcal{B}_m$ is the limit of the inductive family*

$$(*) \qquad \mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{B}_3 \subset \dots$$

of semisimple finite dimensional subalgebras $\mathcal{B}_m \subset \mathcal{B}$.

Proof. In terms of the basis elements b_g the algebra inclusion $\mathcal{B}_m \subset \mathcal{B}_{m+1}$ is determined by the formula

$$b_g \mapsto b_{i(g)} = \frac{1}{|N_m|} \sum_{h \in N_m} b_{gh}.$$

Here we use the notation

$$i(g) = \frac{1}{|N_m|} \sum_{h \in N_m} gh$$

for the corresponding homomorphism $i :\ \mathbb{C}[G_m] \to \mathbb{C}[G_{m+1}]$. One can readily check that this is indeed a homomorphism of $*$-algebras, and that $\mathcal{B}$ is the union of the subalgebras $\mathcal{B}_m$, $m \geq 1$. Note that the map i does not respect the units in the algebras $\mathcal{B}_m$. As a result, there is no unit in the algebra $\mathcal{B}$. $\square$

4. The branching of characters. In this section we describe the Bratteli diagram of the inductive family $\{\mathcal{B}_m\}$ approximating the algebra $\mathcal{B}$.

Let $\widehat{G}_m$ denote the finite set of equivalence classes of irreducible complex representations of the group G_m. We take $\Gamma_m = \widehat{G}_m$ for the mth level of the branching diagram $\Gamma = \bigcup_{m=0}^{\infty} \Gamma_m$.

Given an irreducible representation $\varphi \in \Gamma_m$, we denote by $\widetilde{\varphi}$ the irreducible representation of the group P_{m+1} in the same Hilbert space lifted from φ via the epimorphism $\pi :\ P_{m+1} \to G_m$. Let

$$\mathrm{Ind}\, \widetilde{\varphi} = \bigoplus_{\psi} \varkappa(\varphi, \psi)\ \psi$$

be the decomposition into irreducibles of the representation $\mathrm{Ind}\, \widetilde{\varphi}$ induced to G_{m+1} by the representation $\widetilde{\varphi}$ of P_{m+1}. We denote by $\varkappa(\varphi, \psi) \in \mathbb{N}$ the multiplicity of ψ in the decomposition.

Definition. Let $\Gamma = \bigcup_{m=0}^{\infty} \Gamma_m$ be the vertex set of a branching diagram, and define the multiplicities of transitions as $\varkappa(\varphi, \psi)$, where $\varphi \in \Gamma_m$ and $\psi \in \Gamma_{m+1}$, for

all $m \geq 0$. This is the Bratteli diagram of the inductive system $(*)$. We shall refer to the branching graph $(\Gamma, \varkappa)$ as the *character branching* of the IP-family.

The character branching describes the restrictions of characters of irreducible representations of the algebras $\mathcal{B}_m$. Choose a representation $\psi \in \Gamma_{m+1}$ and consider the restriction $\chi^\psi(i(g))$ of its character $\chi^\psi(g) = \operatorname{Tr}\psi(g)$ to the elements $g \in G_m$. Then

$$(**) \qquad \frac{1}{|N_m|} \sum_{h \in N_m} \chi^\psi(gh) = \sum_{\varphi \in \Gamma_m} \varkappa(\varphi, \psi) \chi^\varphi(g).$$

Remark. The branching diagram $(\Gamma, \varkappa)$ determines the inductive family of algebras $\{\mathcal{B}_m\}$ up to the dimensions of its irreducible representations. The reason is that we disregard the zero components arising in the decompositions of restricted representations. The limiting algebra $\mathcal{B}$ is determined by $(\Gamma, \varkappa)$ up to stable isomorphism.

Let $\Delta_m \subset \Gamma_m$ denote the set of irreducible representations φ of G_m which contain a nonzero $G(1,m)$-invariant vector. Representations in $\Delta = \bigcup_{m=0}^\infty \Delta_m$ will be referred to as *unipotent* irreducible representations of the group G_m. Note that every nonzero irreducible component of the restriction $\psi(i(\cdot))$ of a unipotent representation $\psi \in \Delta_{m+1}$ is itself a unipotent representation in Δ_m, so that $(\Delta, \varkappa)$ is a branching graph, too. In fact, let σ_m denote the representation of G_m induced by the identity representation of the subgroup $G(1,m)$. (In our Example 3 this is the *flag* representation). The unipotent representations can be alternatively defined as the irreducible components of σ_m. The claim now follows from the observation that σ_m, when extended trivially to P_{m+1} and induced to $G_{m+1} \supset P_{m+1}$, is equivalent to σ_{m+1}. The limiting algebra corresponding to the branching graph $(\Delta, \varkappa)$ is the Hecke algebra of the pair $G \supset B$.

5. The branching of conjugacy classes. There is another branching graph associated with every IP-family, and we proceed with the definition of this graph.

Let us say that elements $g_1, g_2 \in G(1,m) \subset G_m$ are *equivalent*, or *G-conjugate*, if $g_2 = h^{-1} g_1 h$ for some $h \in G_m$. In this case we write $g_1 \sim g_2$.

Given a probability measure M on the group $B = \varprojlim G(1,m)$, we denote by $M_m(g)$ the cylinder probabilities $M_m(g) = M\{\mathrm{Cyl}(g)\}$, $g \in G(1,m)$. The coherence condition reads as

$$M_m(g) = \sum_{h \in N_m} M_{m+1}(gh) \qquad \text{for all } g \in G(1,m).$$

Definition. We say that a probability measure M on the group B is *G-central* if its cylinder probabilities $M_m(g)$ depend only on the G_m-conjugacy class of $g \in G(1,m)$, for all $m \geq 1$. A G-central measure M is *ergodic* if it cannot be represented as a nontrivial mixture of other G-central measures.

Example. The normalized Haar measure of the group B is an example of a G-central measure, for every IP-family.

Problem. Describe all ergodic G-central measures on the basic profinite subgroup B of a given IP-family.

Assume that a character χ of the group G (or, more precisely, of the Bruhat–Schwartz group algebra $\mathcal{B}$) is nonnegative on the subgroup B. Then the identity

$$M_m(g) = \chi(\widetilde{b}_g), \qquad g \in G(1, m),$$

defines a probability distribution on the group $G(1, m)$, for all $m \geq 1$. By the formula $(**)$, the family of distributions $\{M_m\}$ is coherent, hence determines a G-central measure on B.

Example. Consider the IP-family of Example 5, Section 1 assuming for simplicity that the coefficient group H is Abelian. The group G in this example is the wreath product of the group $\mathfrak{S}_\infty$ of all finite permutations of the set $\mathbb{N}$ with coefficients in H. The subgroup B is the subgroup of diagonal matrices, hence it is isomorphic to the infinite direct product $B \cong H^\infty \equiv \prod_1^\infty H$.

In this example the subgroup B is normal in G. It follows that the only unipotent representations are those which are trivial on B. All of them induce the Haar measure on the group B.

By the celebrated de Finetti Theorem, the class of G-central measures on B coincides with that of product measures $M = \prod_1^\infty M_0$, for some probability distribution M_0 on H. For instance, take for M_0 the Haar measure on a subgroup $H_0 \subset H$. Then the corresponding G-central measure is, in a sense, positive definite, though it cannot be represented as a mixture of *characteristic measures* arising from the characters of G, nonnegative on B.

II. THE CHARACTERS OF THE GROUP OF ALMOST TRIANGULAR MATRICES OVER A FINITE FIELD

This paper is a sequel of [VK]. We continue the study of the asymptotic character theory of the general linear group $GL(n,q)$ over a finite field $k=\mathbb{F}_q$ with q elements. Here we focus on unipotent characters, and our goal is basically to describe the asymptotics of character values at unipotent conjugacy classes.

Our approach to the asymptotic character theory of the groups $GL(n,q)$ is governed by the character theory of the infinite-dimensional limiting group $GLU(q)$. A character of this group is entirely determined by a probability measure on the group U of upper unitriangular matrices, and we study the statistical properties of such measures.

The group $GLU(q)$ is similar to the group $GLB(q)$ introduced in [VK]. Both $GLB(q)$ and $GLU(q)$ have much more characters than the group $GL(\infty,q)$ studied by E. Thoma [Th] and H. Skudlarek [Sk]. It is this phenomenon that motivates our way of embedding the group algebras $\mathbb{C}[GL(n,q)]$. Another motivation comes from the fact that the branching of unipotent characters with respect to our embedding is governed by the Young lattice, hence coincides with the branching of irreducible characters of the symmetric groups.

We refer to [LSS] for the character theory of locally semisimple algebras, to [Mac] for the definitions of symmetric functions, and to [Zel] for the representation theory of the groups $GL(n,q)$.

0. Unipotent characters of $GL(n,q)$. Let B be the group of upper triangular matrices, and denote by $\mathcal{F}=GL(n,q)/B$ the flag space. Irreducible characters χ^λ contained in the module $\mathbb{C}[\mathcal{F}]$ are called *unipotent*. They are labeled by Young diagrams with n boxes. The degree of χ^λ is given by the following q-version of the *hook formula*:

$$\chi^\lambda(E)=q^{n(q)}\prod_{j=1}^{n}(q^j-1)\prod_{b\in\lambda}(q^{h(b)}-1)^{-1},$$

where $n(\lambda)=\sum_{k\geq 1}(k-1)\lambda_k$. Note that $\chi^{(n)}$ is the identity character, and $\chi^{(1^n)}$ is known as the Steinberg character.

Let P_μ be a parabolic subgroup of type $\mu\in\mathbb{Y}_n$. We denote by ψ^μ the character of the natural representation in $\mathbb{C}[GL(n,q)/P_\mu]$. The decomposition of ψ^μ into irreducible characters has the form

$$\psi^\mu=\sum_\lambda K_{\lambda\mu}\,\chi^\lambda,$$

where $K_{\lambda\mu}$ are known as the *Kostka numbers*. Note that the coefficients $K_{\lambda\mu}$ do not depend on q.

1. The group $GLU(q)$. Let $k = \mathbb{F}_q$ be the finite field with q elements (the field is known to be unique up to isomorphism). An infinite matrix $g = (g_{ij})_{i,j=1}^{\infty}$ over the field k is called *almost unitriangular* if

(1) there are only finitely many nonzero elements below the main diagonal;
(2) all but finitely many diagonal elements are equal to one.

The group $GLU(q)$ of all invertible almost unitriangular matrices is a locally compact topological group. The group U of upper unitriangular matrices is a compact open subgroup in $GLU(q)$. The topology of U is that of the product of the finite set k over all matrix elements above the diagonal. These matrix elements are independent and uniformly distributed in k with respect to the normalized Haar measure M of U.

2. The Schwartz–Bruhat group algebra. The group algebra $\mathcal{B}$ of locally constant compactly supported functions on the group $GLU(q)$ can be naturally represented as the limit of the inductive family

$$\mathbb{C}[GL_1(k)] \subset \mathbb{C}[GL_2(k)] \subset \ldots \subset \mathbb{C}[GL_n(k)] \subset \ldots \tag{1}$$

of the complex group algebras of the groups $GL_n(k)$. The group $GL_\infty(k) = \varinjlim GL_n(k)$ is a countable dense subgroup in $GLU(q)$. Note that the family (1) is *not* generated by the natural embeddings of the groups $GL_n(k)$.

3. Characteristic measures. Every positive definite class function $\chi : GLU(q) \to \mathbb{C}$ is called a *character* of $GLU(q)$. A character χ is *indecomposable* if every decomposition $\chi = \chi_1 + \chi_2$ in a sum of two characters is trivial in the way that both terms are constant multiples of χ.

We find all indecomposable characters of the group $GLU(q)$. In particular, there is a family of *unipotent* indecomposable characters $\chi^{(\alpha;\beta)}$ labelled by pairs $(\alpha;\beta)$ of nonnegative weakly decreasing sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$ such that $\sum_{k\geq 1} \alpha_k + \sum_{k\geq 1} \beta_k \leq 1$. The space Δ of such sequences is known as the *Thoma simplex*. The parameter space Δ of indecomposable unipotent characters is exactly the same as that of indecomposable characters of the infinite symmetric group. Remark that the group $GL_\infty(k) = \varinjlim GL_n(k)$ studied by E. Thoma has only countably many indecomposable characters.

4. Central distributions of unitriangular matrices. Let $U \subset GLU(q)$ be the subgroup of upper unitriangular matrices in $GLU(q)$. Denote by $g^{(n)}$ the submatrix on the crossing of the first n rows and columns of a matrix $g \in U$, and write $\rho^{(n)}$ for the Young diagram describing the Jordan form of $g^{(n)}$. Denote by U_h the cylinder set of all matrices $g \in U$ with $g^{(n)} = h$, and by $\rho = \rho^{(n)}$ the Jordan partition of h.

A probability measure m on U is called *central* if its cylinder probabilities $m(U_h)$ depend on the Jordan partition ρ of the matrix h only. We denote the cylinder probabilities of a central measure as $m_\rho = m(U_h)$.

We say that a set S of matrices in U is a *tail set* if S contains, with every matrix $g \in S$, all the matrices $g' \in U$ that differ from g in only finitely many matrix elements. A central measure m is called *ergodic* if every measurable tail set has probability 0 or 1.

Let $Q_\rho(\alpha;\beta;t)$ denote the dual Hall–Littlewood (super) symmetric function with parameter t.

Theorem. *For every point $(\alpha;\beta)\in\Delta$ of the Thoma simplex there exists an ergodic central measure $m^{(\alpha;\beta)}$ with cylinder probabilities*

$$m_\rho^{(\alpha;\beta)}=\frac{Q_\rho(\alpha;\beta;t)}{t^{n(\rho)}(1-t)^n}, \tag{2}$$

where $t=1/q$, and $n(\rho)=\sum_{k\geq 1}(k-1)\rho_k$ for a partition $\rho=(\rho_1,\rho_2,\dots)\vdash n$.

Conjecture 1. *The above formula provides all ergodic central measures on U.*

Given a matrix $g\in U$, the sequence $t=(\rho^{(1)},\rho^{(2)},\dots)$ is an infinite Young tableau. Set $r_k(\rho)$ for the length of the kth row of a Young diagram ρ, and $c_k(\rho)$ for the length of the kth column of ρ.

Conjecture 2. *The limits*

$$\lim_{n\to\infty}\frac{r_k(\rho^{(n)})}{n}=\alpha_k,\qquad \lim_{n\to\infty}\frac{c_k(\rho^{(n)})}{n}=\beta_k \tag{3}$$

exist for almost all matrices $g\in U$ with respect to the measure $m^{(\alpha;\beta)}$.

5. Characteristic measures of unipotent characters. Every unipotent character $\chi^{(\alpha;\beta)}$ of the group $GLU(q)$ determines a central probability distribution $M^{(\alpha;\beta)}$ on the group U referred to as its *characteristic measure*. We show that

$$M^{(\alpha;\beta)}(U_h)=r_\rho(\alpha;\beta;t)\equiv t^{-n(\rho)}\sum_{\lambda\vdash n}K_{\lambda\rho}(t)\;s_\lambda(\alpha;\beta), \tag{4}$$

where $t=1/q$, $K_{\lambda\rho}(t)$ is the Kostka–Foulkes matrix, and s_λ is the Schur (super) symmetric function.

In order to identify the characteristic measure $M^{(\alpha;\beta)}$ of a unipotent indecomposable character $\chi^{(\alpha;\beta)}$ as one of the measures $m^{(\alpha;\beta)}$ we need more notation. Given $0<t<1$ and a positive number a, we denote by $a^{(t)}$ the geometric sequence

$$a^{(t)}=(1-t)a,(1-t)ta,(1-t)t^2a,\dots$$

If $\alpha=(\alpha_1,\alpha_2,\dots)$ is a sequence of positive numbers, we denote by $\tilde\alpha$ the sequence obtained from α by rearranging the elements of the sequences $\alpha_1^{(t)},\alpha_2^{(t)},\dots$ in decreasing order.

Theorem. *The characteristic measure $M^{(\alpha;\beta)}$ of a unipotent character $\chi^{(\alpha;\beta)}$ coincides with the central measure $m^{(\tilde\alpha;\beta)}$ corresponding to the point $(\tilde\alpha;\beta)\in\Delta$.*

6. Extending unipotent characters from the subgroup U. Every indecomposable unipotent character $\chi^{(\alpha;\beta)}$ of the group $GLU(q)$ is determined, as a functional on the group algebra $\mathcal{B}$, by a signed Radon measure $\tilde M^{(\alpha;\beta)}$ on $GLU(q)$. Given a matrix $h\in GL_m(k)$, consider the cylinder set $C_h\subset GLU(q)$ of all matrices $g\in GLU(q)$ such that $g^{(m)}=h$ and $g_{ij}=0$ if $i>m$ and $i>j$. The cylinder measures $\tilde M^{(\alpha;\beta)}(C_h)$ depend on the conjugacy class of h in $GL_n(k)$ only. Recall that the conjugacy classes are labelled by the families $\varphi:\mathcal{F}(q)\to\mathbb{Y}$ of Young diagrams such that

$$||\varphi||\equiv\sum_{f\in\mathcal{F}(q)}|\varphi(f)|\,d_f=m. \tag{5}$$

Here $\mathcal{F}(q)$ is the set of irreducible polynomials in $\mathbb{F}_q[t]$ with the exception of the polynomial t, and d_f denotes the degree of f.

We show that the cylinder measures $\tilde M^{(\alpha;\beta)}(C_h)$ can be restored from the cylinder probabilities of the corresponding characteristic measure $M^{(\alpha;\beta)}$.

Theorem. *The cylinder measures* $\tilde{M}^{(\alpha;\beta)}(C_h)$ *can be restored from the cylinder probabilities* (4) *of the corresponding characteristic measure* $M^{(\alpha;\beta)}$ *by the formula*

$$\tilde{M}^{(\alpha;\beta)}(C_h) = \prod_{f \in \mathcal{F}(q)} r_{\varphi(f)}(\alpha^{d_f}; (-1)^{d_f+1}\beta^{d_f}; t^{d_f}). \tag{6}$$

Here φ *is the family of Young diagrams describing the conjugacy class of the matrix* h*, and we define the* d*th power of a sequence* $\alpha = (\alpha_1, \alpha_2, \dots)$ *as* $\alpha^d = (\alpha_1^d, \alpha_2^d, \dots)$.

III. A LAW OF LARGE NUMBERS FOR THE CHARACTERS OF $GL_n(k)$ OVER A FINITE FIELD k

Introduction

Let $k = \mathbb{F}_q$ denote the finite field with q elements, and consider the group $GL_\infty(k)$ of infinite nonsingular matrices over k of the form $g = E + f$, where E is the identity matrix and f is a matrix with finitely many nonzero elements. The characters of the group $GL_\infty(k)$ were studied and classified by E. Thoma and his successors (see [Th, Sk]). Since the group $GL_\infty(k)$ is the inductive limit of the finite linear groups $GL_n(k)$, $n \to \infty$, characters of $GL_\infty(k)$ can be regarded as "coherent" families of characters of irreducible representations of the groups $GL_n(k)$.

In a recent paper [VK] the authors have introduced a new locally compact group, $GLB(k)$, made of all nonsingular infinite matrices over the field k with only finitely many nonzero elements below the main diagonal. In a sense, this group can also be regarded as a limit of the finite linear groups $GL_n(k)$, and its characters are represented by coherent (in a sense to be made precise below) families of irreducible characters of the latter groups. One of the advantages of the group $GLB(k)$, compared with $GL_\infty(k)$, is that it has much bigger amount of characters, and that the formal substitution $q = 1$ admits a natural interpretation in terms of the character theory of the infinite symmetric group $\mathfrak{S}_\infty$.

The present paper is a sequel of [VK]. Here we restrict ourselves to unipotent characters of the groups $GL_n(k)$, and the limits thereof. Basically, our claims are as follows:

(1) Unipotent indecomposable characters $\chi^{(\alpha;\beta)}$ of the group $GLB(k)$ (labelled by elements $(\alpha;\beta)$ of the Thoma simplex, see [VK]) can be treated as (signed) measures on this group.

(2) Every such measure is uniquely determined by its restriction $M^{(\alpha;\beta)}$ to the compact subgroup $U(k) \subset GLB(k)$ of upper triangular matrices with the unit diagonal. The restrictions, referred to as *characteristic measures*, are in fact probability measures on $U(k)$.

(3) Characteristic measures corresponding to different points of the Thoma simplex are mutually singular.

(4) We state a conjectural Law of Large Numbers for the characteristic measures. In order to describe it in detail, we need some notation. Let $M^{(\alpha;\beta)}$ be the characteristic measure corresponding to Thoma parameters $(\alpha;\beta)$. Let $u(n)$ denote the submatrix of a matrix $u \in U(k)$ on the crossing of the first n rows and columns, and let $\lambda(n)$ be the partition of n corresponding to the Jordan block structure of $u(n)$. A matrix $u \in U(k)$ is said to be *regular* if the following limits (called the row and column *frequencies*) exist for all $k = 1, 2, \dots$:

$$\lim_{n\to\infty} \frac{\lambda(n)_k}{n} = \widetilde{\alpha}_k, \tag{1.1}$$

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$$\lim_{n\to\infty} \frac{\lambda'(n)_k}{n} = \widetilde{\beta}_k. \tag{1.2}$$

Conjecture. *Almost all matrices $u \in U(k)$ are regular with respect to every characteristic measure M. Set $t = 1/q$ and assume that $M = M^{(\alpha;\beta)}$. Then the list*

$$\begin{aligned} \{&(1-t)\alpha_1, (1-t)t\alpha_1, (1-t)t^2\alpha_1, \dots, \\ &(1-t)\alpha_2, (1-t)t\alpha_2, (1-t)t^2\alpha_2, \dots, \\ &(1-t)\alpha_3, (1-t)t\alpha_3, (1-t)t^2\alpha_3, \dots\} \end{aligned} \tag{1.3}$$

coincides as a multiset (i.e., up to the order) with the list of row frequencies in (1.1)*. Moreover, $\widetilde{\beta}_k = \beta_k$ for all $k = 1, 2, \dots$.*

Example. In the particular case of $\alpha_1 = 1$, $\alpha_k = 0$ for $k \geq 2$, $\beta_k = 0$ for $k \geq 1$, the row frequencies form a geometric sequence

$$\widetilde{\alpha} = \{(1-t), (1-t)t, (1-t)t^2, \dots\}. \tag{1.4}$$

In this case the conjecture follows from a result obtained by A. Borodin [Bor].

Let M be a probability distribution on the compact group $U(k)$ of unipotent matrices. Following the general terminology of the theory of graded graphs and locally semisimple algebras (cf. [LSS]), we say that the measure M is *central* if the probabilities

$$M_\lambda \stackrel{\text{def}}{=} M\{u \in U(k) : \quad u(n) = v\} \tag{1.5}$$

depend not on the matrix $v \in U_n(k)$ itself but on its Jordan form partition λ only. We reduce the main problem of describing the characteristic measures to the more general problem of classifying all central probability measures on $U(k)$. As usual, every such measure admits a unique presentation as a mixture of ergodic (indecomposable) central measures. The latter can be described in terms of the (dual) Hall–Littlewood symmetric polynomials $Q_\lambda(x;t)$ (see [Mac]).

Conjecture. *Given a real number b, we write $b^{(t)} = \{(1-t)b, t(1-t)b, t^2(1-t)b, \dots\}$ for the associated geometric sequence, and we set $\beta^{(t)} = \{\beta_1^{(t)}, \beta_2^{(t)}, \dots\}$ for a sequence β. Then the formula*

$$M_\lambda = (1-t)^{-n} t^{-n(\lambda)} Q_\lambda(\alpha; \beta^t; t) \tag{1.6}$$

correctly defines the cylinder probabilities (1.5) *of a central measure $M = M^{(\alpha;\beta)}$, where $(\alpha;\beta)$ is a point of the Thoma simplex. The measures $M^{(\alpha;\beta)}$ are ergodic and mutually singular, and every ergodic central measure M coincides with one of the measures $M^{(\alpha;\beta)}$.*

This conjecture is equivalent to the conjecture of the second author stated in [GHL, equation (7.3.3)] (see also [Ker]).

Unipotent representations of the group $GL_n(k)$. An irreducible representation of the group $G_n = GL_n(k)$ (and its character) is called *unipotent* if it is a part of some flag representation (that is, a representation induced by a trivial representation of a parabolic subgroup). Equivalently, one can define unipotent representations as those containing an invariant vector for the Borel subgroup $B_n \subset G_n$ of upper triangular matrices.

The unipotent representations π_λ are labelled by the integer partitions $\lambda \vdash n$ of n (we denote the set of such partitions by $\mathbb{Y}_n$). It is well known that the dimension of π_λ is given by the q-hook formula

$$\dim \pi_\lambda = \frac{(q-1)(q^2-1)\dots(q^n-1)}{\prod_{b\in\lambda}(q^{h(b)}-1)}\, q^{n(\lambda)}.$$

The dimension of the subspace of B_n-invariant vectors does not depend on q and is provided by the ordinary hook formula

$$f_\lambda = \frac{n!}{\prod_{b\in\lambda} h(b)}.$$

Example. The so-called *Steinberg representation* $\pi_{(1^n)}$ is the unipotent representation of G_n corresponding to the one-column partition $\lambda = (1^n)$. Its dimension is

$$\dim \pi_{(1^n)} = q^{n(n-1)/2},$$

and there is only one B_n-invariant vector.

Let $\chi^\lambda(g) = \operatorname{Tr}\pi_\lambda(g)$ denote the character of an irreducible unipotent representation π_λ of the group G_n. There is a simple explicit formula for the values $\chi^\lambda_\rho(q)$ of such characters at unipotent elements $g \in G_n$ of type ρ:

$$\chi^\lambda_\rho(q) = \widetilde{K}_{\lambda\rho}(q) \equiv q^{n(\rho)}\, K_{\lambda\rho}(q^{-1}).$$

Recall that $K_{\lambda\rho}(t)$ is the generalized Kostka matrix, see [Mac, Section III.6]. In particular, all the values $\chi^\lambda_\rho(q)$ are nonnegative.

Alternatively, one can define the character matrix $\{\chi^\lambda_\rho(q)\}$ as the transition matrix between the bases of Schur functions $s_\lambda(x)$ and that of modified Hall polynomials $\widetilde{P}_\rho(x;q) \equiv q^{-n(\rho)} P_\rho(x;1/q)$:

$$s_\lambda(x) = \sum_{\rho\vdash n} \chi^\lambda_\rho(q)\, \widetilde{P}_\rho(x;q).$$

Note the similarity of this formula with the classical Frobenius formula providing the character matrix of the symmetric group.

Representations induced from parabolic subgroups. Given a composition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$, we denote by P_μ the corresponding parabolic subgroup generated by the block diagonal matrices in $G_{\mu_1} \times G_{\mu_2} \times \dots \times G_{\mu_m}$ and by the group B_n. Let σ_μ be the representation of G_n induced by the identity representation of the subgroup $P_\mu \subset G_n$. We shall refer to σ_μ as *induced* representations. Note that

the induced representation σ_μ depends only on the partition corresponding to the composition μ (that is, does not depend on the order of parts). By definition, σ_μ is the representation in the space of flags $H_0 = \{0\} \subset H_1 \subset \ldots \subset H_m = k^n$ with the dimension vector $\big(\dim(H_j) - \dim(H_{j-1})\big)_{j=1}^m = \mu$. The dimension of the induced representation σ_μ equals the Gaussian coefficient:

$$\dim \sigma_\mu = \frac{[n]!}{\prod_{j=1}^m [\mu_j]!},$$

where $[n]! = \prod_{j=1}^n [j]$ and $[j] = (q^j - 1)/(q-1)$.

The important fact is that the multiplicities $K_{\lambda,\mu}$ of unipotent representations in the decomposition of a flag representation,

$$\sigma_\mu = \sum_\lambda K_{\lambda,\mu} \pi_\lambda,$$

do not depend on q, and coincide with the *Kostka numbers* arising in decompositions of induced characters of the symmetric group $\mathfrak{S}_n$.

Let $\psi^\mu(g) = \operatorname{Tr} \sigma_\mu(g)$ denote the character of the induced representation σ_μ. By definition, $\psi^\mu(g)$ is simply the number of fixed points in the space of the representation σ_μ (i.e., of flags of a specified dimension vector μ). The value of the induced character ψ^μ at a unipotent element $g \in G_n$ of type ρ can be written in the form

$$\psi^\mu(g) = \psi^\mu_\rho(q) \equiv \sum_{\lambda \vdash n} K_{\lambda,\mu}\, \widetilde{K}_{\lambda,\rho}(q).$$

This is a polynomial in q.

The characters of unipotent factor representations. We refer to [VK] for the description of characters of general factor representations of the group $GLB(k)$ (more precisely, of the group algebra A of Bruhat–Schwartz functions on this group).

We say that such a character χ is *unipotent*[3] if its restriction $\chi\big|_{A_n}$ to every subalgebra $A_n \cong \mathbb{C}[G_n]$ is a linear combination of only unipotent characters of the group G_n. According to [VK, Section 6], the unipotent characters of factor representations $\chi^{(\alpha;\beta)}$ are labelled by the points $(\alpha;\beta) \in \Delta$ of the Thoma simplex Δ. We shall use two explicit decompositions of these characters: one in terms of irreducible unipotent characters χ^λ,

$$\chi^{(\alpha;\beta)}(a_g) = \sum_{\lambda \vdash n} \chi^\lambda(g)\, s_\lambda(\alpha;\beta),$$

and one in terms of the characters ψ^μ of induced representations,

$$\chi^{(\alpha;\beta)}(a_g) = \sum_{\nu \vdash n} \psi^\nu(g)\, m_\nu(\alpha;\beta).$$

[3] In [VK] we have used the term *principal series* instead.

Induced characters at primary elements. The goal of this section is to describe the value of an induced character ψ^μ, $\mu \vdash n$, at a primary element $g \in G_n$ in terms of its values at unipotent elements. Recall (see [Mac, IV.3]) that the primary conjugacy class corresponding to an irreducible polynomial $f \in k[t]$ of degree $d = d(f)$ is characterized by an integer partition $\rho \vdash m$, $m = n/d$. If $d = 1$ and $f = t - 1$, the class is said to be unipotent.

Lemma. *The value $\psi^\mu(g)$ of the character ψ^μ at a primary element $g \in G_n$ with the characteristic polynomial $f^n(t)$, $d(f) = 1$, does not depend on the choice of f.*

Proof. Let $g \in G_n$ be a primary element with the characteristic polynomial $f^n(t)$ where $f(t) = t - a$. Then $g = aE + n$, where $E \in G_n$ is the identity matrix and $n \in G_n$ is a nilpotent element. Let $h \in G_n$ be another primary element with the same partition $\mu \vdash n$ but a different polynomial $t - b$, $b \in k$. We can assume that $h = bE + n$. Since $g - h = (a - b)E$ is a scalar operator, g and h have identical invariant subspaces $V \subset k^n$. Therefore, g and h have the same number of invariant flags of every dimension type μ and $\psi^\mu(g) = \psi^\mu(h)$. $\square$

We are now in a position to consider primary elements with an irreducible polynomial f of degree $d > 1$. Let $K \cong k^d$ be the extension field of k of degree d. The polynomial f has d distinct roots in K, and we denote by $a \in K$ one of the roots. The operator of multiplication by a in K can also be treated as an operator in k^d. It is well known that this is a primary element of $GL_d(k)$ with the characteristic polynomial f and the integer partition $\mu = (1)$. Every semisimple regular element is conjugate to some multiplication operator in the extension field (see [Mac, VI.3, Example 4]).

More generally, let $g \in GL_m(K)$ be a primary element with a partition $\mu \vdash m$ and a linear polynomial $t - a$ where $a \in K$ generates the extension $K \supset k$. Denote by $\widetilde{g} \in GL_n(k)$ the corresponding operator in k^n, $n = md$. Then $\widetilde{g}$ is also primary with the same integer partition μ. The corresponding polynomial f of degree d can be identified as the irreducible polynomial over k with the root a. Every primary element in $GL_n(k)$ can be obtained in this way from a primary element with a polynomial of degree $d = 1$ over an appropriate extension field.

One can easily describe the invariant subspaces of $\widetilde{g}$ in k^n in terms of those of g in K^m.

Lemma. *Let $H \subset K^m$ be a g-invariant subspace of dimension s over K. Then it is also a $\widetilde{g}$-invariant subspace over k of dimension sd. Moreover, every $\widetilde{g}$-invariant subspace in k^n is in fact a subspace over the extension field K. In particular, its k-dimension is divisible by d.*

Corollary. *Let $g \in GL_n(k)$, $n = md$, be a primary element with partition ρ. Then the value $\psi^\nu(g)$ of an induced character ψ^ν only depends on the degree $d = d(f)$ of the irreducible polynomial f corresponding to g. More precisely,*

$$\psi^\nu(g) = \begin{cases} \psi^\mu_\rho(q^d) & \text{if } \nu = d\mu \text{ for some } \mu \vdash m \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $d\mu$ denotes the partition of $n = md$ obtained from $\mu \vdash m$ by multiplying all its parts by d. The polynomial $\psi^\mu_\rho(q)$ was defined at the end of Section 2.2 as the value of a character on a unipotent element.

An extension formula for unipotent characters of the group $GLB(k)$. Let $\chi^{\alpha;\beta}$ be the unipotent character of the group $GLB(k)$ corresponding to Thoma parameters $(\alpha;\beta)$. According to the formulas of Section 2.3, the value of $\chi^{\alpha;\beta}$ at a unipotent element $g \in GL_n(k)$ with partition ρ can be written in the form

$$\chi^{\alpha;\beta}(g) = r_\rho(\alpha;\beta;t),$$

where $t = 1/q$ and the symmetric function r_ρ is defined by the formula

$$r_\rho(\cdot) = t^{-n(\rho)} \sum_{\lambda \vdash n} K_{\lambda,\rho}(t)\, s_\lambda(\cdot).$$

In this section we find an explicit expression for the value of a unipotent character $\chi^{\alpha;\beta}$ at a general element $g \in GL_n(k)$ in terms of the symmetric functions r_ρ.

Theorem. *Assume that the conjugacy class of $g \in GL_n(k)$ corresponds to a family $\varphi : \mathcal{F} \to \mathbb{Y}$ such that $\sum_f \deg(f)\,|\varphi(f)| = n$ (see* [Mac, IV.2]*). Then*

$$\chi^{\alpha;\beta}(g) = \prod_{f \in \mathcal{F}} r_{\varphi(f)}(\alpha^{d(f)}; -(-\beta)^{d(f)}; t^{d(f)}).$$

Proof. Recall (see [Mac, I.2]) that the monomial symmetric function $m_\lambda(x_1, x_2, \dots)$ is defined as the sum of the monomials $x^\gamma = x_1^{\gamma_1}\, x_2^{\gamma_2} \dots$ where γ runs over all distinct finite permutations of the coordinates of λ. Let π_d denote the endomorphism of the symmetric function algebra Λ defined by $\pi_d(p_m) = p_{md}$, for all $m = 1, 2, \dots$. This is the familiar *plethysm* endomorphism, see [Mac, I.8]. Then it follows immediately from the definitions that $\pi_d(m_\lambda) = m_{d\lambda}$.

Given a symmetric function $u \in \Lambda$, set $U = \pi_d(u)$. By definition, we have $U(x) = u(x^d)$. We shall use the following "supersymmetric" version of this formula:

$$U(\alpha;\beta) = u(\alpha^d; -(-\beta)^d).$$

In fact, it suffices to check the formula for the power sum symmetric functions $u = p_m$. One can easily check that in this case both sides are equal to $\sum \alpha_i^{md} - \sum(-\beta_i)^{md}$.

Let us now prove the theorem in the special case of a primary element g with a polynomial of degree d. By the Lemma of section 2.4,

$$\begin{aligned}
\chi^{\alpha;\beta}(g) &= \sum_{\nu \vdash n} \psi^\nu(g)\, m_\nu(\alpha;\beta) = \\
&= \sum_{\mu \vdash m} \psi^{d\mu}(g)\, m_{d\mu}(\alpha;\beta) = \\
&= \sum_{\mu \vdash m} \psi^\mu_\rho(q^d)\, m_\mu(\alpha^d; -(-\beta)^d) = \\
&= r_\rho(\alpha^d; -(-\beta)^d; t^d).
\end{aligned}$$

The theorem now follows from the Multiplication Theorem, see [VK, Section 7, Theorem 6]. $\square$

Unipotent characters at unipotent classes. Let $\chi = \chi^{(\alpha;\beta)}$ be the unipotent character of the group $GLB(k)$ corresponding to a point $(\alpha;\beta)$ of the Thoma simplex. Let ρ be a Jordan form partition of a matrix $u \in GL_n(k)$. In order to describe the character value $\chi(u)$ we introduce appropriate symmetric functions

$$r_\rho(\alpha;\beta) = t^{-n(\rho)} \sum_{\lambda \vdash n} K_{\lambda,\rho}(t)\, s_\lambda(\alpha;\beta),$$

where s_λ is the Schur function. See [VK] and [Mac, Section III.6] for the definition of the generalized Kostka matrices $K_{\lambda,\rho}(t)$.

Theorem. *The value $\chi^{(\alpha;\beta)}(u)$ of a unipotent character $\chi^{(\alpha;\beta)}$ at a unipotent matrix u with Jordan form partition ρ can be written as*

$$\chi^{(\alpha;\beta)}(u) = r_\rho(\alpha;\beta).$$

These characters of $GLB(k)$ are determined by their values at unipotent classes.

IV. AN OUTLINE OF CONSTRUCTION OF FACTOR REPRESENTATIONS OF THE GROUP $GLB(\mathbb{F}_q)$

1. Introduction. It is well known that indecomposable semifinite characters of the group S_∞, which were described in [FA], correspond to type II_∞ representations of the group algebra $\mathbb{C}(S_\infty)$ (and, consequently, the group itself). Recall that in this case, in general, a character takes infinite values on the group. Since the algebra A (the Bruhat–Schwartz algebra for the group $GLB(\mathbb{F}_q)$, see [VK]) is a two-sided invariant dense subalgebra of the true group algebra $L^1(GLB, \mu)$ of the group $GLB(\mathbb{F}_q)$ with the Haar measure μ, it follows by a general theorem (see, e.g., [HR]) that representations of the algebra A can be extended to the group GLB and, consequently, to the group $GL(\infty)$, which is a subgroup of GLB. But, in general, this extension can have another type; more exactly, if a representation of A is of type II, then its extension can be either of the same type or of type III. It is this possibility that occurs in our case. However, even without referring to the general theory, which, as far as we know, gives no answer to the question how one can find the type of the extension, we can nevertheless give a direct description of some representations of the group corresponding to type II_∞ representations of the algebra A.

We will restrict ourselves to one example, namely, the important class of representations which we call *principal series Grassmanian representations.* The described construction is also of independent interest.

2. The infinite Grassmanian, Schubert cells, and their symbols. In this section we generalize the classical methodology of constructing the cell partition of the Grassmanian of finite-dimensional subspaces into Schubert cells (see, e.g., [Miln, Stan]) to the infinite-dimensional case. More precisely, we construct the partition of the Grassmanian of infinite-dimensional subspaces into Schubert cells.

Consider the Grassmanian $\mathrm{Gr}(V)$ of all subspaces of a vector space V over the finite field $\mathbb{F}_q$, where $V = \lim V_n$, $V_1 \subset V_2 \subset \dots$, $\dim V_n = n$, and endow it with the topology of a $GLB(q)$-homogeneous space. In other words, a neighborhood of a given subspace is the set of subspaces that can be obtained from it by applying elements of the group GLB lying in a given neighborhood of the identity element. It is convenient to choose the fundamental base of neighborhoods of a given subspace $t \in \mathrm{Gr}(V)$ that consists of the sets $\{U_{t,n}\}$, where $U_{t,n}$ is the set of subspaces $s \in \mathrm{Gr}(V)$ that have the same intersection with the coordinate subspace V_n as t and the same dimensions of the intersections with V_m, $m \geq n$, as t, i.e., the set of subspaces such that $s \cap V_n = t \cap V_n$ and $\dim(s \cap V_m) = \dim(t \cap V_m)$, $m \geq n$; we will say that such a subspace s is n-equivalent to t. This topology on the Grassmanian is stronger than the totally disconnected topology. The flag of subspaces $V_1 \subset V_2 \subset \dots$, $\dim V_{n F_q} = n$, $n = 1, 2, \dots$, will be called the *principal flag.*

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Every subspace $E \in \mathrm{Gr}(V)$, $E \subset V$, is uniquely determined by the infinite sequence $E \cap V_i$, $i = 1, 2, \ldots$, of its (finite-dimensional) intersections with the subspaces of the principal flag. Consider the dimensions $\dim(E \cap V_i) = d_i$; the sequence $\epsilon_i = d_i - d_{i-1}$, $i = 1, 2, \ldots$, of the differences of neighboring dimensions, which are equal to 0 or 1, is the *Schubert symbol* of the subspace E (it is convenient for us to slightly change the traditional terminology). The subspaces with the same Schubert symbol generate a *Schubert cell*, which, as it is easy to see, is compact and is a B-orbit, where B is the compact Borel subgroup of the group GLB, i.e., the group of upper triangular matrices. Thus the Grassmanian $\mathrm{Gr}(V)$ is a fibration over the space of $(0, 1)$-sequences, with Schubert cells as fibers. Every fiber (cell) can be identified with an affine space of some dimension over the field $\mathbb{F}_q$. The dimension of a cell e is equal to $\dim(e) = \sum_{i=1}^{\infty} i\epsilon_i$. Cells of infinite dimension consist of infinite-dimensional subspaces. We are most interested in cells that consist of subspaces of infinite dimension and infinite codimension.

Every Schubert cell contains a unique coordinate subspace, which can be used as a distinguished point of this cell. Given a $(0, 1)$-sequence ϵ, denote by S_ϵ the Schubert cell with symbol ϵ, and by t_ϵ the distinguished (coordinate) subspace that belongs to this cell. Denote by $\mathrm{Gr}_0(V)$ the set of coordinate subspaces and identify it with the space $(0, 1)^\infty \equiv \Sigma$ of all $(0, 1)$-sequences by the formula

$$\epsilon \mapsto t_\epsilon.$$

Thus the set of $(0, 1)$-sequences parameterizes simultaneously Schubert symbols, coordinate subspaces, and Schubert cells.

Since the group B is compact and all Schubert cells are transitive B-spaces, on every Schubert cell there is a unique B-invariant probability measure, namely, the image of the Haar measure on B. Denote it by M_ϵ.

The orbit of a Schubert cell S with respect to the action of the countable group $GL(\infty)$ is the countable union of Schubert cells over all symbols that differ from the symbol of S in finitely many coordinates and have the same number of 1's in any sufficiently large initial part of the symbol. Now the orbit is a GLB-invariant set, because the group GLB is generated by the subgroups B and $GL(\infty)$. The partition of the space of symbols into the classes of symbols belonging to the same orbit is exactly the partition τ of the space $\Sigma = (0, 1)^\infty$ into the orbits of the group S_∞; it is a tame (hyperfinite) partition. Two subspaces $E, E' \in \mathrm{Gr}(V)$ that belong to Schubert cells of the same orbit (in other words, whose symbols eventually coincide) will be called *congruent*.

On the orbit of every cell $S_\epsilon = S$ we have the σ-finite GLB-invariant measure μ_S normalized by the condition that the measure of the given cell S_ϵ (or of the given B-orbit) is equal to 1. We will call such measures *elementary invariant measures*. The measures corresponding to different cells of the same orbit differ by a positive factor. More exactly, the following proposition holds.

Proposition. *The elementary invariant measure of a compact set is finite, while the measure of any cylinder in* $\mathrm{Gr}(V)$ *is infinite. Let S and S' be two cells whose symbols ϵ and ϵ' coincide starting from the nth coordinate, and let μ_S and $\mu_{S'}$ be the corresponding elementary measures. Then*

$$\mu_S(C)/\mu_{S'}(C) = q^{\dim(S \cap V_n) - \dim(S' \cap V_n)} = q^{\sum_{i=1}^{\infty} i(\epsilon_i - \epsilon'_i)},$$

where C is an arbitrary compact set. Thus the ratio of the measures of cells is equal to the ratio of the exponentials of their dimensions.

Remark. The last sum is finite because the symbols coincide starting from the nth coordinate. Denote the expression introduced in the proposition above by $c(S, S')$; it is defined on pairs of Schubert cells from the same orbit and, consequently, on pairs of subspaces E, E' belonging to such cells: $c(E, E') = c(S, S')$, $E \in S$, $E' \in S'$.

The elementary σ-finite invariant measures are parameterized by the classes of Schubert cells (or by the elements of the partition τ). The function $c(\cdot, \cdot)$ will be called the *fundamental cocycle* on the Grassmanian $\mathrm{Gr}(V)$. It is also possible to regard elementary invariant measures mentioned above as the images of the Haar measure of the group GLB on orbits of cells; however, we cannot directly project the Haar measure onto an orbit (regarded as a homogenous space), because the stationary (parabolic) subgroup is of infinite Haar measure.

Using this transitive action of the group GLB with the invariant σ-finite measure μ_S, we can, as usual, define a unitary representation of the group GLB in the space $L^2(\mu_S)$. But we will study more complicated *factor representations* of GLB.

Let us define new measures on the Grassmanian. First consider a measure ν on the space of symbols Σ. Taking the direct integral

$$\int M_\epsilon d\nu(\epsilon) \equiv M^\nu$$

over this measure, we obtain a B-invariant measure M^ν on the Grassmanian. Let us find conditions under which this measure is quasi-invariant with respect to the group GLB and central with respect to the algebra A (see [VK]).

Proposition. *The measure M^ν is GLB-quasi-invariant if and only if the measure ν is quasi-invariant with respect to the natural action of the group S_∞ on the space Σ. The action of GLB is ergodic if and only if the measure ν is ergodic with respect to the action of S_∞, or, equivalently, if the tail partition τ is ergodic.*

Having the *congruence relation* on the Grassmanian $\mathrm{Gr}(V)$, we can define the principal groupoid corresponding to this relation (see [Ren]); denote it by R. The measure M^ν determines a measure μ^ν on the groupoid regarded as a subset of $\mathrm{Gr}(V) \times \mathrm{Gr}(V)$ (see [Ren]). Let $H = L^2(R, \mu^\nu)$ be the Hilbert space of complex square-integrable (with respect to this measure) functions on the groupoid. According to general constructions (see [LSS]), unitary representations of the groups $GL(\infty)$ and GLB (and, consequently, $*$-representations of the algebra A) are defined in this Hilbert space with the help of a left (π_l) and a right (π_r) actions, respectively.

First consider the representation of the algebra A in the space H. Let $a_g \in A$; then $\pi_l(a_g)$ is an operator that is defined by a kernel and acts as the operator of left multiplication by a function f_g on R.

Now let us introduce an ergodic measure on the set of Schubert symbols that is invariant with respect to the partition τ; according to the remark above, such a measure must be S_∞-invariant, and, consequently, by the well-known de Finetti theorem, it is a Bernoulli measure with parameters $(\alpha, 1-\alpha)$, $0 \le \alpha \le 1$; denote it by μ^α.

Consider the direct integral of the B-invariant measures on the Schubert cells with respect to this measure μ^α; by definition, this is a Bernoulli measure on the Grassmanian; we denote it by μ_α. This measure is B-invariant and GLB-quasi-invariant by definition.

The orbits of the group GLB coincide with the unions of Schubert cells mentioned above. Let us define the Radon–Nikodym (R-N) cocycle of the measure μ_α with respect to the group GLB. It is more convenient to regard the cocycle as a function of pairs of points E and E' on the same orbit. Let E and E' be two subspaces belonging to the same Schubert cell; this means that there exists an element $g \in GLB$ that sends E to E'. Then, by definition, the value of the cocycle at the pair E, E' is equal to the value of the density $d\mu_\alpha(gE)/d\mu_\alpha$. The symbols $\{\epsilon_i(E) = \epsilon_i\}$ and $\{\epsilon_i(E') = \epsilon'_i\}$ coincide starting from some coordinate with index $n(E, E') = n$, and the numbers of 1's among the coordinates of these symbols with indices less than n are the same and equal to $k(E, E') = k$. Since the action of the Borel subgroup preserves the measure and since this action is transitive on the cells, the value of the cocycle depends on the cell only, i.e., the cocycle is a function of symbols and can be calculated on the coordinate subspaces.

Proposition. *Let e and e' be the intersections of the subspaces E and E', respectively, with V_n, and let their dimensions be equal to k. Let $\sigma(e)$ and $\sigma(e')$ be their Schubert cells in the Grassmanian $\mathrm{Gr}_k(n)$. Then the R-N-cocycle of the measure μ_α with respect to the action of the group GLB is equal to*

$$c(E, E') = q^{\dim(\sigma(e)) - \dim(\sigma(e'))} = q^{\sum_{i=1}^{\infty} i(\epsilon_i - \epsilon'_i)},$$

where $\dim$ *is the dimension of a finite-dimensional Schubert cell regarded as a manifold over the field $\mathbb{F}_q$.*

Corollary. *For any n, the restriction of the cocycle c to the subgroup $GL_n(\infty)$ is cohomologous to one. At the same time, the cocycle c is not cohomologous to zero on the whole group GLB, hence there is no (finite or σ-finite) GLB-invariant measure equivalent to μ_α.*

Proposition. *The action of the group GLB on the space* $\mathrm{Gr}(V)$ *with the quasi-invariant measure μ_α is ergodic.*

Proof. Since the action of the group B is transitive, it suffices to make sure that the action of a subgroup of $GLB(\infty)$ on the space $(\mathrm{Gr}_0(V), \mu^\alpha)$ of coordinate subspaces is ergodic. Consider the subgroup S_∞ of $GLB(q)$; its action is ergodic on $(\mathrm{Gr}_0(V), \mu^\alpha)$, because it coincides with the action of S_∞ on the space Σ of all $(0, 1)$-sequences with a Bernoulli measure, which is ergodic by the well-known zero or one law.

Remark. For every n there exists a $GL_n(\infty)$-invariant measure that is equivalent to μ_α.

3. Several remarks on the realization of principal series factor representations of the groups $GLB(q)$ and $GL(\infty, q)$. A brief sketch of the construction of factor representations is as follows.

The construction is based on the same idea that was used in our papers [VK81, VK82], namely, the idea of trajectory (or orbit, or groupoid) model of representations, which goes back to von Neumann. But this construction uses the orbit

equivalence relation, obtained from the orbit partition, instead of the action of groups. We start with introducing a $GLB(q)$-quasi-invariant and B-invariant measure on the Grassmanian $\mathrm{Gr}(V)$, and then, using the orbit equivalence relation on $\mathrm{Gr}(V)$ and this measure, we construct the Hilbert space L^2 in which the required factor representation of the group $GLB(q)$ is realized in a natural way.

Consider the $GLB(q)$-orbit equivalence relation τ on $(\mathrm{Gr}(V), \mu_\alpha)$ and construct the Hilbert space $L^2(\tau)$ (see [VK81]). In this space we have a left and a right unitary representations of the group $GLB(q)$ and, consequently, of the algebra A. It turns out that these representations of $GLB(q)$ are of type III, while their restrictions to the algebra A are of type II_∞; the characters coincide with the characters described in the previous draft. It seems that the phenomenon of changing the type of factors when extending a representation as was mentioned above had not been considered in detail in representation theory.

4. Realization of factor representations related to the Grassmanian. The goal of this section is to describe the factor representation of the algebra A that corresponds to a principal series character χ_α with two frequencies $\alpha = (\alpha_1, \alpha_2)$. These representations are constructed from a measure on the Grassmanian. The construction below is slightly different from the construction in Sec. 3 of this draft.

We start from a graph interpretation of the *mixed grassmanian* Gr of all possible k-linear subspaces of the space V. Given a subspace $E \in \mathrm{Gr}$, denote by $E_n = E \bigcap V_n$ its intersection with the nth space V_n of the principal flag (see above). For $n = 1, 2, \dots$, set $d_n(E) = \dim E_n - \dim E_{n-1}$. The sequence $d_n = d_n(E)$ consists of 0's and 1's and determines a path in the Pascal triangle. We draw this graph in the SE quadrant. The value $d_n = 0$ corresponds to a horizontal edge (in this case $E_n = E_{n-1}$), and the value $d_n = 1$ corresponds to a vertical edge (the dimension of E_n/E_{n-1} is equal to 1). If the subspace E_n is generated by the subspace E_{n-1} and the coordinate vector e_n, then the extension is called a *coordinate extension*. Of course, besides the coordinate extension, there are $q^m - 1$ other extensions, where m is the codimension of E_{n-1} in V_{n-1} (as well as the codimension of E_n in V_n). With each such extension we associate the vertical edge between the corresponding vertices of the Pascal triangle.

The *Pascal q-triangle* is the branching graph $P(q)$ whose vertices and edges are the same as in the ordinary Pascal triangle $P = P_1$, but the vertical edges $(i-1, j) \to (i, j)$ from the jth column are assigned the multiplicity q^j. The dimensions in the graph $P(q)$ are nothing else than the Gaussian binomial coefficients, and the paths of length n from the initial vertex parameterize our subspaces $E_n \subset V_n$.

The Grassmanian Gr is identified with the space of infinite paths in the (multi)graph $P(q)$ (recall that a path is a sequence of edges). The group $GLB(q)$ acts in a natural way on V and, consequently, on $\mathrm{Gr}(V)$. Orbits of the Borel group B (i.e., Schubert cells, see above) correspond to *plaits*, i.e., sets of paths with the same vertices. Every plait contains a distinguished path corresponding to the sequence of coordinate subspaces. The set of distinguished paths determines a subgraph of $P(q)$ which is the ordinary Pascal triangle P, so we have the inclusion $P(1) \subset P(q)$.

Let us say that a subspace E is *almost coordinate* if almost all extensions $E_{n-1} \subset E_n$ are coordinate extensions; let Gr_n be the subset of subspaces $E \subset V$ that are coordinate starting from some index n. Let $\mathrm{Gr}_\infty \bigcup \mathrm{Gr}_n$ be the set of all almost coordinate subspaces $E \subset V$.

We endow the space Gr_n with a finite measure M_n which is defined as the

average of the Bernoulli measure with parameters (α_1, α_2) on the set of coordinate subspaces with respect to the action of the finite group G_n permuting the first n edges of paths in the graph $P(q)$. The direct description of this measure is as follows. Let each vertical edge have the weight α_1 and each horizontal edge have the weight α_2. Then the M_n-measure of the cylinder of paths with common initial part t is equal to the product of the weights of the edges that belong to t. The full measure of the space Gr_n is equal to

$$M_n(\mathrm{Gr}_n) = \sum_{m=0}^{n} \binom{n}{m}_q \alpha_1^m \alpha_2^{n-m},$$

where $\binom{n}{m}_q$ are the Gaussian binomial coefficients. When we extend the Grassmanian Gr_n to Gr_{n+1}, the measure M_{n+1} extends the measure M_n. Thus we have a σ-finite infinite measure M on the space $\mathrm{Gr}_\infty = \mathrm{Gr}(V)$. This measure coincides with one of the measures constructed in the previous section; it possesses invariant and quasi-invariant properties with respect to the groups B, $GL(\infty)$, GLB and, consequently, determines the corresponding factor representations.

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